7 God and Set Theory.

I am making the claim that God is known by thought. The invisible things of God are perceived, being understood by the things that are made. The question becomes, how does this work? How do we see with the eye of reason? Basically we make the idea of God plausible. That is, given what we know, we attempt to establish the idea of God as consistent with and expressible in the language and concepts that we have. I am treating God as a discernible but hidden object. That is, God is discernible from other objects and is describable by means of the knowledge of the creation. God is conceivable but this does prove that God exists. We do not prove existence. Proving existence is ontological thinking: using thought to obtain existence. Rather we indicate existence by means of thought. But to indicate existence by thought, so that we know where to look, is science.

A good example of this use of thought to locate existence, is the supposed existence of black holes. As explained by Stephen Hawking in A Brief History of Time (Hawking (1990)) the idea of a stellar object having so great a gravitational field that no light could escape from it, was proposed in 1783 by John Michell (ibid, p86). In this area, theory has been guiding experimentation for many years. Hawking says ‘Black holes are one of only a fairly small number of cases in the history of science in which a theory was developed in great detail as a mathematical model before there was any evidence from observations that it was correct’ (ibid, p97). Generally speaking, thought, in the form of mathematics, has been structuring and directing ideas about what may exist. That which can be given a mathematical form has plausibility in the culture. Although this by no means proves that something exists, mathematical thinking structures and clarifies our thinking. I seek to apply mathematical structures and concepts to God. However, we commence with ideas of God and see if they are amenable to mathematical thinking.

7.1 Ideas about God

We have various descriptions, or descriptors, of God. God may be described as wise, beautiful, good, holy, right and just. These are words that are generally used and have no particular reference to God. When these words are used God is not usually being referred to. However, these words can be used in two ways as descriptors of God. One way is to push the idea contained in the word (concept) as far as it can go. This means that God is the greatest form of goodness or beauty that is conceivable under these terms. The other way is to collect these different terms together. This means that God can be referred to by the greatest collection of disparate terms. Therefore, God is described by the greatest collection of maximized terms that is possible. Very few objects could be described by the six descriptors above, let alone in their maximized form. It is important to recognise that not all terms or descriptors apply to God. This is because God is a discernible (by thought) object and therefore can be, and needs to be, differentiated from other objects. God has the internal logic of description that any object has. This means that there are things that God is and things that God is not.

The six descriptors used above have their genesis in the creation. Various things can be described as good, right or beautiful. This means that we have a root idea in the concept of the descriptor. That which is beautiful commences with the idea of beauty. To use an idea from Set Theory, Beauty is well-ordered. That is, there is a least idea of what beauty is. A well-ordering implies that I have a least term with an inductive algorithm or recursion that can generate the sequence of terms required. With respect to the attribute of beauty, I have a least term (or definition) of beauty and a method of generating a sequence of ideas that express increasing ideas of beauty. Beauty may be compared with ugliness, but I claim that beauty has an initial idea that is 'beauty' and not something else, such as the least ugliness. Beauty may be understood or explained as the absence of ugliness. But ugliness is about ugliness and beauty is about beauty, although they can be compared with each other. This well-ordering becomes important when the descriptor of beauty is maximized, that is, when the attempt is made to say that one thing is more beautiful than another and to describe the most beautiful. If beauty is seen on a scale of beauty, we commence with the initial or seed idea of beauty (as distinct from any other descriptor) and proceed to maximize the application of the term. This leads to the idea of a thought sequence that, commencing with the initial or least term of the concept of beauty, develops the extent and applicability of the term beauty until we have the ability to describe the most beautiful thing.

The question arises as to whether there are measurable degrees of beauty. Beauty in an object may be a measure of an object, so that one measurable object is more aesthetically pleasing than another measurable object. For instance, the Golden Rectangle in architecture has a more pleasing effect that other geometric shapes. These more pleasing shapes are measurable, but the real sense of beauty may be in the response evoked by the object. After all, beauty is in the eye of the beholder.

This discussion is about the plausibility of a thought sequence developed by a descriptor of God such as beauty. I wish to relate degrees of attributes of God to Anselm's Definition of God. I claim that Anselm's Definition (God is that than which nothing greater can be thought) introduces the idea of a thought sequence in our attempt to locate God. I am suggesting that Anselm's Definition is saying that God can be conceptually located (understood) by means of a thought sequence. I will relate this thought sequence to an ordinal sequence. But before I do that, I will describe how a sequence with respect to beauty (as an example) would be possible. The idea is to start with a minimally beautiful thing and maximize the beauty of that thing. This will be done both as an external measure of the beautiful object and as an internal measure of the degree to which the object is pleasing.

Consider a computer screen. A face appears on the screen. This can be a randomly generated face, or else a face that is minimal with respect to beauty. But it has to be a face and not a collection of things that could make up a face. (We need the well-ordering of an initial face.) Commencing with this face, incremental (measurable and countable) changes are made to it, either by a person or by the machine, until the decision is made that a maximally beautiful face is obtained. This is an attempt to measure the beauty of an object as measured by the degree of pleasure in a person viewing the object. The basic point is that beauty can be measured in a sequence of responses to a stimulus of beauty. A more beautiful face (as measured by the evocation of pleasure in the viewer) is incrementally approached. Alternatively, an attempt may be made to measure the face on the screen as an objective measure of beauty. Suppose that the most beautiful face is the face obtained as the mode (most frequent measure) of the measure of facial features. Let this be done by adjusting facial pictures or outlines to the same scale and then superimposing the facial outlines on one another on the screen. This can be done in any order. As this is done, common facial features are superimposed and become highlighted. As the sample size is increased, a facial outline begins to appear which is the assembling of the most common facial features for that group. Let us say that the terminating face (highlighted by the common facial features) is the most beautiful for that group. What we now have is an approach to beauty by a photo sequence and an objective measure of beauty for that group. What the above two examples are attempting to do is to show that a thought sequence along the attribute of beauty is conceivable.

7.2 Anselm's Definition as a Thought Sequence

It is my intention to recast Anselm's Definition so that its mathematical and non-metaphysical structure may be seen. I will be attempting to show that Anselm's Definition, stripped of its traditional metaphysical language can plausibly take on a mathematical meaning and accommodate mathematical terminology. This is intended as an example mathematical thinking structuring thought about God.

7.3 Sequence and Limit

Anselm's Definition may be viewed as an increasing sequence with a limit.

An increasing sequence may be defined as a function f of a single variable on the natural numbers such that $f(x) \geq f(y)$ whenever $x > y$. A limit is a point or number approached by an increasing sequence of numbers or points.

With respect to Anselm's Definition, the limit will be the limit of an increasing infinite sequence. By means of an infinite increasing sequence a limit is approached as closely as desired. This is usually written as follows (Horadam (1968), p21).

$f(x)$ approaches a limit $L = f(y)$ for all $y > N$
if for each $\varepsilon > 0$ there exists a $\delta > 0$
such that $|f(x)-f(y)| < \varepsilon$ for all $|x-y| < \delta$

For Anselm's definition, $f(x)$ is the thought sequence and $L=f(y)$ is the limit thought denoting God. The rest of the language above gives an accurate meaning to 'approach'.

The advantage with this definition, for Anselm, is that it is constructive and the sequence is never completed. With respect to Anselm's Definition, this means that whatever thought of God is produced another, greater in the attribute being considered, can be produced. The limit is only approached. Cauchy (whose definition this is) was part of the move to arithmeticize Calculus so that there is no reference to infinitesimals or ‘spirits of departed quantities’. There is no use in this definition of completed infinity, only the unending sequence of potential infinity.

However, the phrase ‘in the limit’ states that the limit is reached. For an increasing infinite sequence, the limit is (usually) not an element of the sequence but the sequence does indicate a limit.

In the context of Anselm's Definition, it is important to see that the limit is not arrived at from within the sequence. The nature of Anselm's Definition is that whatever thought is produced one can always produce a greater in some measure of that attribute. This is the nature of the knowledge of God that Anselm wants to use. No matter what God-specific idea one uses to describe God, God is that being that can handle the greatest extent of any God attribute. (I say God-specific, because not every idea (say evil) ends up with God.) Anselm needs an infinite sequence so that he can include any extent of a divine attribute. Therefore the limit cannot be obtained from within the sequence. It is possible to have infinite sequences that contain their limit. Consider the sequence $\{x_n\}$ where $f(x_i)= a$ (a constant) for all x. However, this is not a sequence that models Anselm's Definition. Such a sequence may model an attribute of God that does not involve a sequence. For instance, one may argue that when

Right is applied to God as an attribute, God is altogether Right and not partially Right. That is, one cannot have measures of Right with respect to God. Therefore, we can say that God is always Right (as the value of the constant), no matter what condition applies. However, we can also apply the attribute of Right to God by commencing with the least (minimal) concept of Right, which becomes the beginning of the sequence. A minimal concept of Right does not apply to God (as fully understood) nor is it a true measure of the Rightness that is found in the idea of God that Anselm is promoting. Therefore, one proceeds on a conceivable sequence, which ends up being infinite by the nature of the subject, namely God.

The > Relation.

Anselm's Definition refers to thoughts that are greater than other thoughts. To Anselm, thoughts that are greater than other thoughts get the thought sequence going. In Set Theory the < relation, and conversely the > relation, (strictly) order a partially ordered set. The order is one of precedence, whereby one term (thought) of a sequence is judged to be prior to (by some rule of precedence) another term (thought) of the sequence. For Anselm's Definition, this depends on one thought being considered as greater than (or less than) another thought. It is conceivable that one thought can be greater than another on a scale of significance or descriptive power with respect to an attribute. I have illustrated above, how the attribute of Beauty may be treated if we push the idea of ‘greater than’ beyond the purely intuitive level of degrees of significance with respect to an attribute. But, even if I am not permitted the use of scaled attributes to illustrate Anselm's Definition, then I would argue that attributes in a maximal form (the greatest beauty or goodness) can be grouped in a maximal collection across applicable attributes. And yet in Anselm’s argument, even for maximal groupings of maximal attributes as a descriptor of God, we have still not reached the limit description of God.

I do not wish to describe the > relation, as related to Anselm's Definition, as an operator. The > relation recognises a situation, it does not create the situation. (The > relation recognises a relation between two terms that is irreflexive, anti-symmetric and transitive.) It is not the > relation that creates the gradations of the attribute, but the > relation is a way of recognising a sequence and placing in an order the sequence of ideas about an attribute. Operators, operating on sets such as the power set operator, are able to generate sets as in the construction of the cumulative hierarchy.

In Anselm’s Definition, we have an increasing infinite sequence of thoughts governed by the > relation. This sequence has a limit, which is not an element of the thought sequence. However, the limit is a thought and the limit denotes God. That is, God is not beauty or goodness but God is known by beauty-in-the-limit and goodness-in-the-limit. These are thought sequences that lead us to the idea of God.

As a geometric illustration consider the circle defined as the limit position of an n-sided regular polygon. As the number of sides is increased and the length of each side is decreased we have an increase in the number of smaller equal sides. As this process proceeds the regular polygon approaches the shape of a circle. There is a sequence of polygons getting closer and closer (arbitrarily close) to a circle. When does the polygon become a circle? – in the limit. However, the circle is never a polygon so it is not an element of the sequence of polygons. There is a qualitative difference between a circle and a polygon, although they can appear very similar. The many-shaped sequence of polygons leads to the single shape of the circle, just as the many attributes of God lead to the one concept of the supreme Being.

Relating this again to Anselm's Definition, the polygon sequence corresponds to the thought sequence. The circle as the limit position corresponds to the limit thought and denotes God, who is not a thought.

Here it may be noted that the circle or the limit position is itself an object (thought) but not an object of the sequence. In general, any attribute of God considered as a thought sequence can have a sequence of greater thoughts generated from it. It is conceivable that there are greater and greater ideas of goodness and holiness (or separation to God) all of which are used to approach the God Anselm is seeking to describe.

The > relation between orderable ideas, can also be replaced by an inclusion of sets of ideas when we consider Anselm's Definition as an ordinal sequence (which we do below). Is the > relation some kind of operator? I would not argue that the > relation is or masks an operation. If, in a hierarchy of sets, one set is generated from another set, the operator (such as a power set operation) needs to be specified. The > relation describes the order resulting from the operation.

Initially, I have sought to establish the mathematical nature of Anselm’s definition. Here mathematics is being used to clarify a philosophical argument.

I am claiming that Anselm’s Definition models the terms of an infinite number sequence (with a limit) as follows.

- The number sequence corresponds to the thought sequence. Presumably each thought could be given a number which specifies its order in the thought sequence.
- The number limit of the infinite number sequence corresponds to the limit thought of the infinite thought sequence, denoting God.
- The limit of the number sequence, which is not a term of the number sequence itself, corresponds to the assertion that the limit thought is not a thought of the thought sequence. This means that the limit thought is the thought beyond the thought sequence and is therefore the appropriate thought to denote God.
- That the limit exists and is unique, corresponds to the existence and uniqueness of the limit thought denoting God. I do not claim that every infinite thought sequence ends up with God. It depends upon which initial thought is being used (we do not obtain God from evil). But I do argue that, given the attributes of God, when these are

maximized, the thought of God is the unique object of thought specified.

- The thought sequence can be expressed as a recursion.

I am not arguing for the Cartesian ontological argument, and I believe Anselm can be reinterpreted avoiding this traditional attack also. This may be seen as follows. Anselm's Definition is attribute free. Anselm's Definition can be stated without any reference to attributes. The argument does not depend on a particular attribute such as perfection. In Anselm's Definition, attributes are instances, or illustrations, of what Anselm wants to say. The power of the Definition is that it is an algorithm, or method, which is an example of a way of thinking that will yield a particular maximized thought. This also emphasizes its mathematical character.

This recasting of Anselm's Definition is to show the logical plausibility and cultural adaptability of his definition. What leads to God and even asserts God does not guarantee God. God is the goal of the conceivability sequence but in the argument God always lies, ultimately, outside the conception. God is conceivable by means of the creation, but ultimately what we can say about God is just our knowledge of God, and this does not give us God, in the sense of proving God’s existence. I would add that knowledge about God can be correct knowledge and can be indicating the correct object. That is, God is conceivable and we are being pointed in the right direction.

7.4 Anselm's Definition as an Ordinal Sequence

Cantor devised two number concepts from a well-ordered collection C of numbers. The first number concept was the ordinal number (ord(C)). By this Cantor meant that he abstracted from the set the nature of the elements but retained their order. The second number concept was the cardinal number (card(C). By this Cantor meant that he abstracted both the nature and order of the elements in the set C leaving only the count of how many elements there were in the set.

Cantor, proceeding to a completed infinity, called the first infinite cardinal $\aleph_0$ and the first infinite ordinal ω. From this Cantor developed transfinite arithmetic. However, in locating mathematical analogues for Anselm's Definition there is no need to go beyond the first transfinite number, because this is as far as I need to go on the basis of Anselm's Definition. $\aleph_0$ is also the cardinality of the set N of natural numbers. The infinite set of integers is called countable and any set in 1-1 correspondence with N is called countably infinite or denumerable.

Von Neumann defined ordinals as follows.

$$
\begin{array}{ll}
0 = \varnothing & (\varnothing = \text{null set}) \\
1 = \{\varnothing\} = \{0\} & \\
2 = \{\varnothing,\{\varnothing\}\} = \{0, 1\} & \\
3 = \{\varnothing,\{\varnothing\},\{\varnothing,\{\varnothing\}\}\} = \{0, 1, 2\} & \\
. & . \\
. & . \\
. & .
\end{array}
$$

That is, the increasing sequence of ordinals is such that any ordinal is the collection of all previous ordinals. The ordinals are ordered by inclusion so that (if ord(x) is the ordinal x) then ord(0) ⊂ ord(1) ⊂ ord(2) ⊂ ord(3)… Generally for any ordinal α, $\alpha_{n+1} = \alpha_n \cup \{\alpha_n\}$.

This relates to Anselm’s Definition in that the thought sequence can be regarded as an ordinal sequence whose elements are ordered by inclusion. Each thought in the sequence is contained in or subsumed by the following thought. Beauty considered as a thought sequence can be ordered by the inclusion relation ⊂. This would mean that any greater thought of beauty would include a lesser thought of beauty. This implies that gradations of beauty subsume or include earlier forms of beauty. In that something is more beautiful than something else, then that beauty includes any previous beauty. This would apply generally to attributes or descriptors of God such as goodness, love, or peace.

So following the ordinal model, the thought sequence is ordered by inclusion and is increasing ultimately towards an ordinal limit.

7.5 Well-ordering

Another feature of the ordinal sequence is well-ordering. A well-ordered set W is a set where every subset of W has a unique first element.

The significance of this minimal or first element in a well-ordered ordinal number sequence (set), is that the thought process does not go infinitely in a reverse direction, into an infinite descent of thoughts. There is a minimal attribute description, which commences the thought process. The thought process has a greatest lower bound. The thought sequence is isomorphic to the ordinal sequence up to ω.

The mechanism for developing the sequence can be the ordinal sequence as described above, the power set operator as described in the cumulative hierarchy or the recursion formula.

7.6 Anselm's Definition and the Limit Ordinal

Having developed the idea that Anselm's Definition is a thought sequence and this being likened to a number sequence approaching a limit, I now want to develop the similarity of the thought sequence to the use of the limit of an ordinal number sequence. Rather than use the greater than relation > we shall use the inclusion relation ⊂. If a and b are thoughts then a ⊂ b means thought a is included in thought b. That is, what is stated in thought a is implied in thought b, but what is stated in thought b implies more than is stated in thought a.

There are advantages in this development. Firstly, the inclusion relation is more suggestive of greater thoughts including lesser thoughts in the thought sequence. Secondly, it can be shown that there are limit ordinals. Thirdly, the particular definition of an ordinal limit point as having no predecessor, fits in well with the limit thought of the thought sequence. Fourthly, the first infinite ordinal ω suggests the concept of God as ω-conceivable. Fifthly, the inclusion relation suggests an inclusive hierarchy of languages used to describe the thought sequence.

I proceed now to show that an ordinal sequence can be shown to have a limit. In the ordinal sequence there are ordinals called limit ordinals. There are two kinds of ordinal. One kind is an ordinal with an immediate predecessor such as 3, $\omega + 1$, $\omega.2 + 5$. The other kind is an ordinal which has no immediate predecessor such as 0, ω, $\omega.2$. Such an ordinal is called a limit ordinal. ω is the first infinite limit ordinal.

We will build a picture of an ordinal limit to demonstrate the use of the limit concept with ordinals (following Sierpinski, p287 and Rotman and Kneebone, p87).

This discussion is about transfinite ordinal numbers in general. By treating Anselm's thought sequence as an ordinal sequence we show how it can be demonstrated that the thought sequence can have a limit.

An λ-sequence is a sequence of ordinals α_ξ such that $\alpha_\xi{:}\xi < \lambda$. This means a sequence of ordinals for ξ less than the limit number λ. Such a sequence has a limit α if for every $\beta < \alpha$ there is a $\mu < \lambda$ such that $\beta < \alpha_\xi < \alpha$ whenever $\mu < \xi < \lambda$. This can also be written as $\alpha = \lim \alpha_\xi{:}\xi < \lambda$. When the λ-sequence α_ξ has a limit this is the same as the union of all previous ordinals $\cup\alpha_\xi{:}\xi < \lambda$. Hence it is uniquely defined. We now wish to show that for any limit number λ every increasing λ-sequence $\alpha_\xi{:}\xi \leq \lambda$ has a limit. A limit number is a limit with no immediate predecessor. This can be done as follows.

Since λ is a limit number, the set of ordinals up to λ has no greatest element. Therefore we can put $\alpha = \sup\{\alpha_\xi : \xi < \lambda\}$ which says that α is the supremum or least upper bound of the sequence α_ξ. This fits the requirements for the definition of a limit for a λ–sequence given above for $\alpha_\xi <$ α for values of $\xi < \lambda$. Therefore $\alpha = \lim \alpha_\xi : \xi < \lambda$. That is, generally for (transfinite) ordinals the limit can be shown to exist.

Having established the importance of the limit point, or number of the ordinal sequence, it is important to realise its significance for Anselm's Definition. It is important to see that the limit thought of God has no immediate predecessor. When any particular thought in the sequence is specified one can also generate a greater thought. This infinite thought sequence must be allowed to continue so that in the conclusion to the sequence no particular thought of the sequence is being referred to. The limit thought is not a thought of the sequence. Therefore it can denote God and not be treated as a continuation of the 'greater than' thought sequence.

It must be pointed out that more ordinal numbers greater than ω can be generated. These are called the Transfinite ordinals. However, these are not part of Anselm's Definition. ω is important as a finite or conceivable form of the infinite (a completed infinity). This original use by Cantor to conceive of the infinite as finite (the potential as actual) suits Anselm's Definition whereby the infinite thought sequence can have a finite concept as a limit. I will call this 'God as finite but known in the limit'. So I am not denying further transfinite uses of the ordinals but I am saying that they are not required within the confines of Anselm's Definition.

This leads to the idea of God as ω-conceivable. That is, God is conceivable by means of the concept of ω.

Anselm's Definition requires a completed infinite set. This is because the Definition is an attempt to locate God. God is approached by the thought sequence but there is always the intention to find God. Therefore we need the combination of the infinite search and the completed find. Now ω, being a number without a predecessor, allows us to place the thought denoting God beyond the sequence but indicated by it. God as ω-conceivable is the finite beyond the infinite process, the conceivable result beyond the inconceivably finished process. This expresses the import of Anselm's Definition.

Lastly using the ordinal sequence, we can speak of a hierarchy of languages used by the thought sequence. The initial thought of the well-ordered thought sequence will be a thought expressed in a first-order language. A second order language gives us the ability to speak about the objects in the first order language. A second order language permits quantification over classes as well as individuals. An n-order language permits discussion of concepts, sets or individuals at any previous level, including discussion of its own individuals. If a class of individuals becomes a concept or property, then orders of languages permit discussion of concepts of concepts of concepts … of some attribute. There is a distinction here between using a language at a particular conceptual level and simply talking about particular degrees of difference such as degrees of beauty, which may still be at an individual level or first-order level. If I want to talk about an attribute at a particular point in the thought sequence, a hierarchy of languages guarantees that I can do so. Any given level of language may have its own individuals. So any thought sequence can be expressed in an inclusive order of languages. If we have an ω^{th} order language (a limit point language) we then have a language to describe that which is denoted by the limit-thought, namely God. Such a language would have concepts (sets) and relations between completed infinite sets, which would be words denoting concepts as descriptor limits such as limit-goodness or limit-beauty.

7.7 Anselm's Definition and The Constructible Hierarchy

When we consider the thought process, which is implied in Anselm's Definition, we can treat this process simply as greater and greater thoughts along some attribute (as an illustration of the process). It is necessary to examine ways of generating the thought sequence. A mathematical analogue of this is the cumulative hierarchy.

Gödel has introduced constructibility in set theory. This is defined as follows (after Devlin (Barwise, p454)). The universe V of all sets is obtained by commencing with the null set, $\emptyset$. So the first set is $V_o = \emptyset$.

Using the power set operation, the number of sets is iterated. The power set is the set made up of all the subsets of a set. If a set N has n elements then N has 2^n subsets. Applying the power set operator to a set V_β we add on the number of subsets obtained from V_β to get the total number of subsets so far. That is, for any ordinal α the total number of sets ν_α so far is

$$V_\alpha = \cup\{P(\nu_\beta):\beta<\alpha\}$$

where P is the power set operator.

This says that the collection of sets in the universe (labeled V_α) is the sum of all previous subsets obtained by the power set operator. This is the cumulative hierarchy.

Generally this can be written as

$$V = \cup_{\alpha\in On} V_\alpha$$

which says the Universe of sets V is simply the addition of all previous subsets where On is the class of all ordinals and V is the class of all sets.

This represents the summation of every conceivable set up to an ordinal limit. But On (as the class of all ordinals) is transfinite which implies that the power set operator is operating on infinite sets. What this could mean is not clear, because we are talking about an operation on an infinite set which, by definition as a construction, cannot be completed. This is also a reason for not using transfinite number with respect the Anselm's Definition. The construction of a thought sequence is necessary for Anselm's Definition so that we can actually approach the limit thought of God. If we place ourselves in the un-constructible (transfinite), whereby we are claiming that infinite sets are constructible in some sense, then we will be in difficulty.

This concept of the cumulative hierarchy gives us a framework for the thought sequence of Anselm. However, the power set operator generates every conceivable subset of a set. Analogously, this says we generate every conceivable thought associated with some attribute. There needs to be some constraint on this process. In set theory the constructible hierarchy has been devised to limit this proliferation of subsets in the cumulative hierarchy. The method used in set theory is to introduce a function Def(X) which accepts for the next level of the constructible hierarchy only those subsets defined in first-order logic over the structure of a given language L. The language L consists of specified relations, functions and constants defined for that structure. In this way a control can be maintained over the development of sets and analogously, of the thought sequence. So the constructible hierarchy is a defined restriction of the cumulative hierarchy.

So we continue. Let X be any set. By Def(X) is meant all the subsets a of X which are first-order definable in L.

By recursion on the ordinals the constructible hierarchy is defined as

$$L_o = \varnothing$$

$$L_{\alpha+1} = \mathrm{Def}(L_\alpha)$$

$$L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha \quad \text{if } \lambda \text{ is a limit number.}$$

The constructible universe is the class $L = \cup_{\alpha\in On} L_\alpha$.

We have demonstrated a mechanism for set generation under constraint and analogously a mechanism for thought generation under the constraint of defined functions and relations for Anselm's thought sequence.

This construction can also be used to generate a recursive hierarchy of languages where each level in the hierarchy represents a thought language. Hence the thought sequence is based upon a language sequence. This is specific evidence for the constructibility of Anselm’s thought sequence.