12 God and Paradox.

12.1 Anselm's Definition and Contradiction.

Central to Diagonalisation and Gödel’s Incompleteness Theorem is the use of Contradiction, which is an element of paradox. Contradiction is expressed at the beginning of this thesis in the initial idea, expressed by Paul, that the invisible things of God are made visible by the things that are made. Because God cannot be observed in the usual sense of observing things, statements about God will hit the objection of how can you see what cannot be (usually) seen? I claim that God exists and therefore God is not only conceivable. God is conceivable but how do we get from God as conceivable to God as existent? This is the thrust of ontological argument. I am claiming that one cannot go from thought to existence, to obtain existence from thought, because existence is not thought (or theory) driven. Existence is the result of the creative act of God. What exists is arbitrary, in that what exists (physically) depends entirely on God. What exists will be conceivable and indicated (or described) by thought, but not determined by thought. Thought (expressed in language) is reflection on what exists. Thought is possible because knowledge is obtained from the properties of objects and the relations between them.

There is a basic contradiction between potential and actual infinity. Potential infinity is the description of the infinity process from within the sequence. By definition there is always another term of the sequence. Constructively, one never arrives at the limit of the sequence. Actual or completed infinity says that we can complete an infinite sequence. This is the view from outside the sequence. This paradox in the ways of speaking about infinity comes out specifically in Anselm's Definition. According to Anselm, God is what you can never reach by thought, but you use thought to reach (some idea of) God. On the one hand you start with the idea of God (what Anselm calls Faith seeking Understanding) and then you use thought to attempt to reach what you know you cannot reach. But Anselm's Definition claims that if you do this you are going in the right direction.

Consider the elements of Anselm's Definition,

1. Thought sequence,
2. Limit thought not reached by the sequence but supplied as a concept,
3. God is beyond what can be thought.

What is the use of thought here? I claim that thought is evidence, pointing us in the direction of God. But how can we explain what we cannot think? What is there conceivable about the inconceivable? What is there describable about the indescribable? This is the problem for religious language and imagery with the added problem of when is religious language metaphor, simile or direct description. So paradox or contradiction is an important element of Anselm's Definition.

I propose to list briefly the epistemological or semantic paradoxes aware of an intriguing comment by Gödel that ‘Every epistemological antinomy can likewise be used for a similar undecidability proof’ (Gödel (1992), footnote 14, p40). Following this I will describe four extended examples of paradox and relate these to Anselm's Definition.

I will now outline some epistemological or semantic paradoxes showing the key ingredients of self-reference and negation.

12.2 Varieties of paradox

The Liar Paradox

This sentence is false.

The paradox: if the sentence is true it is false,

if the sentence is false it is true.

Russell’s Paradox

This paradox deals with the problem of whether every problem determines a set.

Let $A = \{x: x \notin x\}$.

That is, let A be the set of all sets that are not members of themselves. Is A a member of itself?

The paradox: A is a member of itself only if it is not a member of itself and if A is not a member of itself it is a member of itself.

Berry’s Paradox

“the least integer not definable in less than 19 syllables” is defined in less than 19 syllables.

The paradox: the definition in quotes defines a number requiring at least 19 syllables for its definition but it does so in 18 syllables.

Grelling's Paradox

A word is Autological if it describes itself.

A word is Heterological if it does not describe itself.

Which is heterological?

The paradox: if heterological describes itself it is autological. If heterological is heterological it does not describe itself which is describing itself.

Richard's Paradox

Let the set E be the collection of all numbers designated by a finite number of words. All finite word descriptions of numbers can be ordered lexicographically forming a denumerable set. The numbers described can also be listed so that corresponding to every number is a number description. This produces a matrix of number descriptions.

Diagonalize the number matrix producing a number not on the list. Hence we have a new number description not on the number description list.

The paradox is that initially all finite number descriptions were created and listed. But we have shown how to generate a number description not on that list.

Gödel in his paper compares his argument to Richard's paradox (ibid, p40) and the Liar.

The features of Richard's Paradox that are used in Gödel's argument are the coding of formulae as numbers and the creation of a proposition that asserts its own unprovability.

12.3 The Ingredients of Paradox

A paradox is 'a seemingly sound piece of reasoning based on seemingly true assumptions that leads to a contradiction or false conclusion' (Audi (1996), p558). Statements are contradictory if one is the denial of the other.

Consider the following sentences.

1. This sentence is short.
2. This sentence is not short.
3. This sentence is false.

Sentences 1 and 2 are unproblematic.

Sentence 3 is the Liar, which is very problematic. This is because of the combination of self-reference, negation and truth claim. Consider the difficulties associated with the following statements.

<u>No self-reference.</u>

If we say 'that sentence is not true', there is no problem.

<u>Self-reference and negation</u>

'This sentence is not long' contains self-reference and negation but is unproblematic.

<u>Self-reference and truth claim</u>

‘This sentence is true’ is unproblematic. It is self-referential but not contradictory.

Self-reference, negation of truth claim

‘This sentence is not true’ is a paradox.

A paradox is an argument producing a contradiction. One can either accept the contradiction (paraconsistency) or reject the argument generating the contradiction. Both Anselm and Gödel argue by contradiction. Anselm’s argument by contradiction is his second argument (see 4.4).

12.4 Uses of Contradiction

Anselm gets his result by contradiction. That is, God, who is beyond thought, is arrived at by thought. This contradiction is brought out particularly in Diagonalisation, where by means of a given set of decidable predicates, indicated by a binary matrix, one can always obtain an anti-diagonal. In the three examples that follow, the first example is the contradiction obtained by Gödel where he isolates truth from provability. The second example from Mostowski gives his version of Richard’s paradox, followed by an example using contradiction to obtain unprovability in a formula. The third example is an interesting version of Gödel’s Incompleteness Result by Boolos. All of these examples can be reworked as a form of diagonalisation (see 10.3).

Consider the following uses of Contradiction.

1. The first example of the use of contradiction is Gödel’s preliminary explanation of his theorem. (This is more fully explained in 11.1 and 10.3 (6).)

Consider a formula list $R_i(n)$.

Isolate $R_n(n)$.

Define $n \in K \equiv \neg Bew(R_n(n)))$, ($\neg$Bew means not provable).

But $n \in K$ is some formula $R_i(n)$ in the list say $R_q(n)$.

So $R_q(n) = n \in K \equiv \neg Bew(R_n(n))$.

For n = q we have

$$R_q(q) = q \in K \equiv \neg Bew(R_q(q))$$

which says

$R_q(q)$ is provable implies $q \in K$ which implies $R_q(q)$ which is not provable. Negating both sides gives

$R_q(q)$ is not provable implies $q \in K$ which implies $R_q(q)$ which is provable.

So $R_q(q)$ is undecidable.

2. The second example of the use of contradiction is Mostowski's version of Richard's Paradox (Mostowski, p5)

Let $S = \{W_1\ W_2\ W_3\ \ldots\}$ be the sorted list of all descriptors of properties of all integers which is countably infinite.

$Prov(W_p(n))$	says 'integer n has property p' is provable.
$\neg Prov(W_p(n))$	says 'integer n has property p' is not provable.

Now $\neg Prov(W_n(x))$ describes n so it must be a property on the list say $W_q(n)$.

Then putting n = q gives

$Prov(W_q(q)) = \neg Prov(W_q(q))$ which is a contradiction.

$W_q(q)$ is both provable and not provable. So $W_q(q)$ is not provable in S. Mostowski adds that unprovability is a metalanguage concept so how do we express $W_n(n)$ as not provable in S? This is the problem Gödel solved with arithmeticisation.

Mostowski continues (Mostowski (1964), p8) with another example of the use of contradiction saying Gödel says (in effect),

Let $\phi(n,p)$ be the Gödel number of $W_p(n)$ and let T be the set of Gödel numbers of theorems of the formal system S.

But $W_n(n) = \phi(n,n)$ is not a theorem (provable) in S (from the previous result above).

That is, $\phi(n,n) \notin T$.

But $\phi(n,n) \notin T$ is a theorem of S. It is equivalent to saying that $W_n(n)$ is not provable in S. Hence $\phi(n,n)\notin T$ is some W_q in S. Putting n=q we get $W_q(q)$ is $\phi(q,q)\notin T$, which is a theorem stating it is not a theorem.

3. The third example of the use of contradiction is a Boolos version of Gödel's Incompleteness Theorem (Boolos (1999), p387).
Consider a machine M that only prints out all true and only true statements.
Let A(x,y) say 'x applies to y' and x is printed out as a true statement. A(x,y) is a propositional function that takes the value 'true' if x can be related to y in some unspecified way.
Let n be the Gödel number of $\neg A(x, y)$ that is, 'x does

not apply to y'.

Assert that 'n applies to n' (i.e. A(n,n)). If this is true, n is printed out which says n does not apply to n. So the printout says n does not apply to n which contradicts n does apply to n, so n is not printed out to avoid contradiction. So the true n (¬A(n,n)) is not printed out. So we have a condition where a truth is not printed out.

12.5 Infinity and Contradiction.

Dummett, in his book Elements of Intuitionism (Dummett (1977)) contrasts potential and actual infinity.

> The Platonistic conception of an infinite structure as something which may be regarded both extensionally, that is, as the outcome of a process, and as a whole, that is, as if the process were completed, thus rests on a straight contradiction (ibid, p41).

The concept of actual or completed infinity, or even infinities (which suggests that infinite processes as finalities may be compared as in Cantor's Transfinite) is rejected by Intuitionism (as explained by Dummett). Because of the constructivist nature of Intuitionism, one cannot speak of a completed infinite process that cannot, by definition, be completed. However, the sequences 1,2,3,… and 2,4,6,… both have the same cardinality $\aleph_0$, which is the cardinality of the set of natural numbers. The two different processes have the same number of elements which is not intuitively obvious, as the process, 2,4,6,… does not reach **N** twice as quickly as 1,2,3,…. This is the problem one gets when one concentrates on the process or one regards the process as the only source of information. Intuitionists want an infinite process. Dummett even says ‘that there is no objection to introducing into the language of intuitionistic mathematics genuinely denotative symbols, such as ω, for quantities which, like the denumerable ordinals, stand in no finite ratio to positive, finite numbers’ (ibid, p40). But ω denotes a completed (infinite) process, principally because the process is being viewed as a single object or event. The nub of the contradiction is a sequential, continuous process being portrayed as a completed, discrete event. The continuous process becomes a completed object of thought and it can be treated both as a process and as a (single) product. To speak of infinity is not only to label a process but it is also to name an event where the process is collapsed into a conceptual unity. I claim that this conceptual unity, or concept, is necessary for Anselm's Definition and for general mathematical results.

What does one do with a sequence that does not terminate? If the sequence does not terminate one is always dealing with a multiplicity of terms not all of which can be known and whose properties one can never fully be described. Even in the Principle of Induction on **N**, one cannot say ‘for every n ∈ **N**’ without surmising a general behaviour of all terms in the sequence, constructed or non-constructible. The constructivist is continually relativised to and bound by the sequence. On the integers there are infinitely many sequences but only one infinity. In fact there are more sets or sequences of integers than integers. The Intuitionist is left comparing sequences, not being able to distinguish different types of infinite totalities. A particular sequence can refer to only some elements of **N**, those that can be constructed. Whereas classical mathematics can refer to any element of **N**, constructible or not. The theory of the infinite is not about the constructible but about the conceivable, which may not be constructible. The Intuitionist, of course, feels that they have been rescued from the 'nonsense' of interminable and inaccessible transfinite cardinalities. Because completed infinite sequences can themselves becomes terms of further completed infinite sequences, the process appears interminable and obviously not able to be constructed. However, the reaction against Cantor's 'heaven' can impoverish mathematics and remove what I regard as a necessary element of Anselm's Definition.

The constructivist, like the reductionist, is the one who, in terms of Anselm's Definition, remains in the sequence. I do not deny that there is a genuine problem here. It is true that 'to grasp an infinite structure is to grasp the process that generates it' (ibid, p40) because the finite, describable process indicates a described, infinite result. I claim that there is a reduction here from concept to process that impoverishes thought. There is a way to speak conceivably about the completed infinite and Cantor has shown us how to do it.

In terms of Anselm's Definition, there is the necessity of the completed infinite, because God is the object beyond the thought sequence. To claim that there is nothing beyond the sequence is to make God a thought and to remain forever in the sequence. If there is no completed sequence, then

God is denoted only by a term in the sequence. This implies that any term can always be extended so that the greater than is not greater than any other term. The fact of the contradiction between the potential and actual infinite is essential to Anselm's Definition. Because Anselm's Definition uses both the constructible thought sequence and the object beyond, or indicated by, any particular term of the sequence, both infinities are required with the inherent contradiction between them.