11 God and Incompleteness

11.1 Gödel's Incompleteness Theorems.

I have attempted to show so far that Anselm's Definition can be considered as a sequence with a limit, with the elucidation of associated ordinal properties. I have also considered Anselm's Definition in the light of Church's Thesis showing how the attempt to formalize an algorithm is similar to Anselm's attempt to formalize thought about God. In looking at Diagonalisation I have sought to show the means of constructing the logical possibility of an uncreated object.

In this chapter I seek to show how Anselm's Definition can be related to Gödel's Incompleteness results. Principally this will attempt to show that, if a formalised consistent system has an undecidable truth, the idea of God can be regarded as truth beyond the provability of a thought sequence. Although this is clearly an analogy, by pursuing Gödel's argument, I hope to go beyond analogy and give conceptual evidence for the plausibility of the existence (truth) of God. I regard conceptual evidence as stronger than analogy (see 6.1).

I will now look at Gödel's Incompleteness Theorem and indicate ways by which Anselm's Definition may be illustrated, elucidated and given plausibility.

In his paper (Gödel (1992), p38), Gödel argues as follows.

Consider an ordered list R of all class signs for a formal system (as per Richard's Paradox). A class sign is a formula with one free variable. The notation is as follows:

(I will vary Gödel's notation so that $R_n(x)$ uses a subscript n for the n^{th} position in the list R.)

$R_n(x)$ is the n^{th} class sign in the list with one variable x.

$R_n(n)$ means the class sign variable takes the position of the class sign in the list.

Bew x says x is provable and ¬Bew x says x is not provable.

The argument is as follows:

Consider $n \in K \equiv \neg Bew[R_n(n)]$ which says n is a member of K (K a set of natural numbers) if n, substituted in R for x, is not proven.

$\neg Bew(R_n(n))$ is a class sign therefore in the list R.

$n \in K$ is also a class sign in R, say S.

Suppose S is the q^{th} class sign in R. That is $S = R_q(q)$.

Therefore $S = R_q(q) \equiv q \in K$.

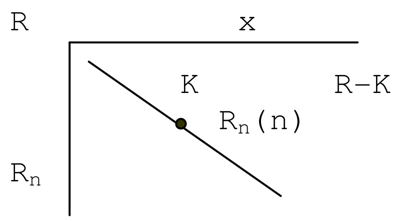
Therefore $R_q(q) = q \in K \equiv \neg Bew[R_q(q)]$. Negating these formulae gives $\neg(R_q(q)) = q \notin K \equiv Bew[R_q(q)]$.

Gödel now has his proof of undecidability as follows:

Suppose $R_q(q)$ is provable	Suppose $\neg(R_q(q))$ is provable
Therefore $R_q(q)$ is true and	then $\neg(R_q(q))$ is true and
$q \in K$ which implies	$q \notin K$ which implies
$\neg Bew\ R_q(q)$ which says	$Bew\ R_q(q)$ which says
$R_q(q)$ is not provable	$R_q(q)$ is provable
– contradiction	– contradiction

This proves that the formula $R_n(n)$ is undecidable in PM (which stands for Principia Mathematica which is the formal system Gödel chose as the system within which to work). That is, it cannot be proved or disproved. This is the core of Gödel’s argument. The work in Gödel’s result is actually constructing, recursively, a formula that is true but cannot be either proved or disproved.

This argument by Gödel can also be treated as a diagonalisation as follows: Let $R_n(x)$ be the class sign or formula of one variable in a list having the x^{th} column on the n^{th} row. Then $R_n(n)$ is a formula on the diagonal of the list.



We are considering the formulae $n \in K \equiv \neg Bew[R_n(n)]$ which says n is a member of K if $R_n(n)$ is unproven. $R_n(n)$ indicates the diagonal among the formulae. Let the diagonal be the set of natural numbers K when $R_n(n)$ is not proven. The set of formulae R (the matrix list) without K, that is (R-K), is the set of formulae that are proven.

Now if $q \in K$ (that is, q is on the diagonal) then $R_q(q)$ can be interpreted as being not proven. However, if the diagonal as a constructible formula is treated as another formula for the list, then it is proven.

Hence we obtain a result where the same formula can be treated as proven and unproven or undecidable.

Anselm's argument from contradiction can be summarised as follows:

> If that which is <u>greater than</u>
> is not treated as greater than
> then the <u>greater than</u> is not greater than
> – which is a contradiction.

This is a parallel argument for Gödel's Theorem. Whereas the above argument used the <u>greater than</u> as an object and greater than (>) as a relation or predicate, in Gödel's Theorem one can use what is <u>proven</u> (as object) and what is unproven (as relation) as follows.

> If that which is <u>proven</u>
> is treated as not proven
> then that which is <u>proven</u> is not proven
> – which is a contradiction.

This is an interpretation of the line in Gödel's sketch of a proof

$$R_q(q) = q \in K \equiv \neg Bew\ [R_q(q)].$$

Both of these forms of their respective arguments contain some of the ingredients of paradox: self-reference and negation.

Hilbert intended to create a vigorous and finitist foundation for mathematics so that every mathematical fact could be put on a sure axiomatic foundation. By the technique of axioms and transformation rules acting on axioms <u>all true</u> mathematical statements were to be derived. There were two criteria for such an idealistic system: the final mathematical result was to be consistent and complete. These two technical terms had the following meaning:

> Consistency was obtained if the system S did not prove for any $p \in S$, both p and not-p. That is $\neg(p \wedge \neg p)$.
>
> Completeness was obtained if the system S could prove for any $p \in S$ either p or not-p.

Gödel was able to develop formulae in a system PM (Principia Mathematica) which were rich enough to express arithmetic. Gödel was able to show that a formula (17 Gen r) was not provable and its negation (¬(17 Gen r)) was also not provable. Hence he called the formula and the system undecidable. This was a major blow to Hilbert's programme, although some have argued that there are various interpretations of the implications of Gödel's result (Shankar (1988)).

11.2 Summary of Gödel's Incompleteness Theorem

The following is a summary of Gödel's 1931 paper (Gödel (1992)).

1. Introduction

   $n \in K \equiv \neg Bew[R_n(n)]$

   Here Gödel likens his argument to Richard's paradox and the Liar paradox (ibid.,p. 40). Gödel also makes the point

   > "The method of proof just exhibited can clearly be applied to every formal system having the following features
   >
   > 1. Interpreted as to content it provides for a coding of 'provable formula'.

2. Every provable formula is correct ‘as regards content’.”(ibid, p41)

2. The formulation of a formal system ‘for which we seek to demonstrate the existence of undecidable propositions’. This is done by

Basic signs (symbols)	Represented by	Numbers
Series of basic signs (formulae)	Represented by	Numbers
Series of series of basic signs (proofs)	Represented by	Numbers

This representation by numbers is the Gödel numbering technique whereby symbols, formulae and proofs are expressed as unique numbers (products of powers of primes).

3. Recursion

Using strings (which can be expressed as numbers) Gödel builds up metamathematical concepts from primitive symbols by computational procedures. This is done by Recursion.

A formula derived as a consequence of construction by recursion is valid in the system in which it is constructed. Gödel will proceed to recursively derive a formula that is valid by construction but which will assert its own unprovability. Gödel chose Principia Mathematica (PM) as the level of formalism he would use. In this formal system PM, there is a 1-1 correspondence between the formulae of the system and Gödel numbers, where a Gödel number can uniquely represent symbols, formulae and proofs. This means that the metamathematical concepts such as formula, axiom immediate consequence or proof can be represented and constructed as strings of numbers.

4. Demonstration of Undecidability
   By recursion, Gödel builds the string ‘17 Gen r’ or ‘v Gen r’, where ‘v Gen r’ is the generalisation of r with respect to its free variable v. Gödel is able to show that both ‘17 Gen r’ and ‘Not(17 Gen r)’ are unprovable within PM. So, in a system that is meant to be consistent (that is, one cannot have p and not(p)), Gödel has demonstrated the construction of an undecidable formula.

5. Gödel goes on to show that the sentence stating the consistency of the system also cannot be proven. This result was an immediate obstacle for Hilbert’s programme which was to establish the consistent provability of all of arithmetic.
   As an example of the kind of reasoning involved, consider the following (after Braithwaite (ibid, p25)) using a formal system P.

   Gödel claims in his Proposition XI (ibid, p70), that if P is consistent, then the statement saying P is consistent is not provable within P. The point of the following argument is to show how the consistency of P cannot be proven. It is done by showing how a formula, which is logically equivalent to the statement of the consistency of P, states its own unprovability.

Let y, a particular formula, be provable in P.
If not-y is provable in P, then P is inconsistent.
If not-y is not provable in P, then P is consistent.
That is, the consistency of P is logically equivalent to the unprovability of not-y, for all provable y.
What is required is a recursive formula in P expressing the unprovability of not-y. Consider

x B y	this says x proves y. (This is a recursive relation sign in two free variables x,y.),
not(x B not-y)	this says x does not prove not-y. (This is a recursive class sign in one free variable x.),
x Gen(not(x B not-y))	this says for every x, x does not prove not-y. (This is a recursive formula in no free variables. Therefore it is a statement.)

All the terms of this last formula can be recursively generated and the meaning of the formula expresses the unprovability of not-y. Therefore, if the consistency of P (the formal system), is logically equivalent to the unprovability of not-y (for all provable y), we have demonstrated how a recursive statement can state the unprovability of not-y and hence the unprovability of the statement of the consistency of P. Hence consistency and unprovability are related.

11.3 The Finsler Version of Undecidability

In Van Heijenoort (Van Heijenoort (2000)) a paper by Finsler sketches an interesting presentation of the problem of formal proofs and decidability. The paper, as summarised by van Heijenoort (Heijenoort, p438), clearly shows the difference between truth requirements and provability. It is important for Gödel to avoid the paradox of the Liar so Gödel’s result is an argument about provability in formal systems not an argument about truth in formal systems, although the result may be related to truth claims.

Consider the lexicographical list developed in Richard’s paradox (see chapter 11). Let $F_i(j)$ be the i^{th} formula with one free variable j.

Consider the binary matrix a_{ij}

0 1 1 0

a_{ij}

where 1 says $F_i(j)$ is true

0 says $F_i(j)$ is false.

Consider another binary matrix b_{ij}

1 0 1 0

b_{ij}

where 1 says $F_i(j)$ is provable

0 says $F_i(j)$ is not provable.

The anti-diagonals a'_{ii}, b'_{ii} for each matrix are formed

$1 - a_{ii} = a'_{ii}$

$1 - b_{ii} = b'_{ii}$.

This changes each 0 to 1 and 1 to 0.

In diagonalisation the anti-diagonal is not any row of the matrix. That is, there is no k (row) such that for every j (column)

$$1 - a_{jj} = a_{kj}.$$

If there were such a k then putting k = j gives

$$1 - a_{kk} = a_{kk}$$

which is Richard's paradox.

However, although the anti-diagonal a'_{jj} of a_{ij} is not equal to a row of a_{ij} it is possible that a'_{jj} may be equal to a row of b_{ij}. That is

$$a'_{jj} = 1-a_{jj} = b_{kj}.$$

If this is true, and remembering the a_{ij} and b_{ij} have only binary values (only 0,1) we can only have $b_{kj} = 0$ when $a_{jj} = 1$ or $b_{kj} = 1$ when $a_{jj} = 0$. That is, considering values for

$$1 - a_{jj} = b_{kj},$$

we get option a) $1 - 0 = 1$

or option b) $1 - 1 = 0$.

Option a) says when $a_{jj} =0$ (is false), $b_{kj} =1$ (is provable).

Option b) says when $a_{jj} =1$ (is true), $b_{kj} =0$ (is not provable).

A condition on Gödel's result is that the formal system be consistent (that is all formulae are true). So option a) (where the formula a_{jj} is false) cannot apply. So option b) is the required result. Option b) says that you can have a consistent formal system of true formulae but there is the condition of that system that there is a true formula that cannot be proven.

This result ultimately isolates (at least one) truth from provability in a consistent system. This does not mean that true statements cannot be proven, but only that at least one true statement (in a formal system of only true statements) cannot be proven.

11.4 Tarski's Semantic Incompleteness

Popper argues that Tarski's theory of truth has rehabilitated the correspondence theory of truth: that truth is correspondence to the facts (which suited his realist position). It also suits a position consistent with a theory of creation where what is created is objectively real.

Tarski's truth theory claims that an object language L_0 requires a richer semantic language L_1 to make such statements as 'true in L_0'. The higher order language can talk about the lower order language. Tarski established the nature of the higher order language (or meta-language) as having to contain names of statements, statements describing states of affairs in the object language (which can be part of the meta-language) and generally terms of higher order such as properties of properties. Semantic incompleteness claims that there can be no general criterion for truth. There will always be required a semantic meta-language L_{i+1} for language L_i.

As Barrow points out (Barrow (1992), p125) deductive systems, which are formally incomplete, are semantically (or conceptually) incomplete also. This has two repercussions. One repercussion is that reductionist attempts to contain all truths in a derivable conceptual system cannot succeed. Barrow comments (ibid., p144) that logical positivism (which can be likened to Hilbert's ambition to make all mathematics strictly derivable) fails in its attempt to exclude metaphysical (and theological) statements because they are not derived from a verifiable axiomatic system. There is a truth in L_i that can only be explained in L_{i+1}.

The other repercussion is that when I speak of a conceptually complete creation, such a creation is also conceptually (semantically) incomplete. This means that there is a truth required to be expressed in a language of higher order than the language describing the conceptually closed creation. I claim that the statement 'God exists' could be such a truth. This statement meets the requirement of being undecidable (as I have explained earlier). 'Undecidable' means that there is no procedure available to determine if a theorem is proven or not proven in a consistent system. I claim that there is no proof, or disproof, of the statement 'God exists'. I believe that I can argue for God's existence and point out its plausibility and relevance but not prove the fact of existence.

As pointed out, one ramification of this result for philosophy is that philosophical systems, which claim that the only true statements are the provable statements, are faced with the possibility of unprovable true statements. A philosophical reductionism such as empiricism (or materialism) that claims that the only meaningful (true) statements are the observable (empirical or material) statements cannot claim that all true statements of their system are provable (which is usually their criterion for truth). Nor can they say that the claim, that all their true statements are provable, is itself provable. No matter how tight the connection between truth and provability is claimed for a system, there is always at least one true statement not provable in that system.

This most certainly leaves room for Anselm in his thought sequence to make truth claims about God that are true but not provable. Not provable here means undecidable within the system as it stands. A Proof may be obtained by adding axioms but the new system also has a truth not provable in the system at any given time. (Knowledge of) God may be viewed as a concept true and consistent with the thought sequence but conceivable and acceptable only in a limit position, which lies outside the actual thought sequence. On this interpretation, if proof as a deductive system is required, then we must remain in the sequence. This means that the limit can be a truth outside the deductive system.

It is important to note here that I am not attempting to prove that the truth not proven by the system is a true but unproven statement about God. What I seek to maintain is that, if there is a truth about God that is not provable in the system under consideration then this incompleteness result allows for its existence. Using Anselm's Definition, I can develop a thought sequence leading to the limit concept of ‘God’. I claim that there is no difficulty in having thoughts about God. The real question is whether the object described by these thoughts is instantiated? Does this God exist? If I conclude that this Incompleteness result allows for a true but unproven formula, G, within the consistent formulae of the system, can this statement G be the claim that God exists? I cannot prove that G is true, nor can I even prove that G is instantiated, but the Incompleteness result allows for the possibility and plausibility of the claim. I am also claiming that existence cannot be proven. That is, I can demonstrate an existence claim but I cannot prove that what I claim is instantiated. I can argue that horses exist and narwhals (with spiral tusks) exist and therefore it is reasonable that unicorns exist. However, this does not guarantee that the concept 'unicorn' is instantiated. Similarly, the concept 'God' may be rationally derived but this is no proof that the concept 'God' is instantiated.

11.5 The Significance of Gödel's Incompleteness Theorems for Anselm.

Gödel claims that the minimum requirements on a formal system P to obtain his result are that

1. The class of axioms and rules of inference are recursively definable in P and that

2. Every recursive relation is definable in the system P. (ibid, p.62)

When these two conditions apply undecidable propositions exist. Also in every extension of P where axioms are added an undecidable sentence is obtained.

This may be described as

A_1	A_2	A_3	. .	. . . . . .	A_ω	
	T_1	T_2	T_3	. . . . . .	. .	T_ω

Here A_1 is a consistent set of axioms and formulae. The Gödel Incompleteness result states that there is a true but unprovable (undecidable) formula T_1 outside that system. When further axioms A_2 are added to make the system complete (contain all true formula and prove T_1) another unproven truth T_2 lies outside the system, and so on. This sequence of axioms A_{i+1} being added to prove the unproven truths T_i can be continued up to ω, assuming that we have a recursive procedure for generating more axions. In this case T_ω becomes a truth without axioms A_ω, as the latest unproven truth is always without axioms that prove it. Thus T_ω can represent the unproven truth derived from the consistent system of formulae. This truth need not be about God, because the result is purely formal. If, however, we use attributes relating to God, and, after the manner of Anselm's Definition, we develop a thought sequence, then the intention is to produce thoughts leading to the limit thought of God. I am arguing that the idea of God as a limit is conceptual evidence for the existence of God, not proof of the existence of God.

In a footnote (ibid, footnote 48a, p62) Gödel claims that

> the true source of the incompleteness attaching to all formal systems of mathematics is to be found in the fact that the formation of ever higher types can be continued into the transfinite, whereas in every formal system at most denumerably many types may occur. It can be shown, that is,

> that the undecidable propositions here presented always become decidable by the adjunction of suitable higher types (eg of type ω for the system P). A similar result also holds for the axiom system of set theory.

Gödel seems to be arguing that the adjuction (adding) of type ω, the first transfinite ordinal, renders the system P decidable. If we are dealing with a formal system, at most denumerably many types (ordinals) may occur. Therefore, this implies that the ω level inclusion is both true and provable (decidable). This would suggest that, if the ω ordinal limit number formally designates Anselm's ω-thought description of God, 'God' as an idea is at least true and provable. However, I would argue that the addition of ω-axioms to the ω-truth ($A_\omega \cup T_\omega$) would produce another unproven truth. What is the characteristic of infinity that overrules, or suspends, the canons of provability? I suggest there is none.

A result in computability is that the set of computable (programmable) functions is denumerable (Cutland, p77). There are many more functions than computable functions as can be demonstrated by a diagonalisation on the set of functions.

A computable function is a function executable by a machine. The set of machines is denumerable. This is seen by considering a machine as a programmable device such as a Turing machine. The set of programs is effectively denumerable (they can be listed). A formal system is a 'mechanical', axiomatized way of generating formulae.

But what happens in the limit? With a formal system ω is the limit.

The set of true statements of a formal system P is not recursive because it is not decidable by Gödel’s Incompleteness Result. That is, there are true but unprovable statements with respect to any consistent formal system.

Here I am equating a formal system with a mechanical procedure and with a computable function. Therefore, if there are at most $\aleph_0$ computable functions then there are at most $\aleph_0$ formal systems. Because of the denumerable nature of any formal system there is at most $\aleph_0$ formulae in any formal system.

How does this relate to Anselm’s Definition?

The thought sequence can be viewed as a formal system commencing with basic definitive thoughts or axioms and then generating thoughts as formulae creating new combinations or extensions of thought all consistent with rules of inference, or immediate consequence, including substitution of formula in free variable positions.

Gödel claims that condition 1 above (recursive definition) “is in general satisfied by every system whose rules of inference are the usual ones and whose axioms (like those of P) are derived by substitution from a finite number of schemata” (ibid, p.62).

Anselm’s definition, in that it is attribute free, can be treated as a formal system. That is, the mechanism for inference that is specified as a formal requirement is the

$>$ relation and no particular attribute is required. It is important to make the distinction between a relation and an operator in the thought sequence. The > relation compares thoughts and claims that one is greater than the other, in some sense. I have used the example of beauty and claimed that a maximal beauty can be approached. This will allow for maximal beauty within any object and across all possible objects. This allows for optimal beauty in one object as being more beautiful than optimal beauty in another object. (Everything is beautiful in its own way.) The argument is attribute independent and is in that sense formal. However, the generation of a term representing greater beauty is done by an operator, which I have called the GT operator. A thought on which the GT operator will operate can be expressed formally as an axiom or a primitive (symbol) of the system. In the recursive generation of the terms of the thought sequence, we could use the following $m(k+1,x) = GT_B(k, m(k,x),x)$, where m is the recursive function, k is the index counting the steps of the recursion and GT_B is the construction operator generating a term denoting greater beauty.

Alternatively, the rule of inference, or how we obtain one concept from the previous concept, such as the GT operator may be expressed as the power set operator or by substitution of formulae in free variable positions.

These ideas are suggesting that Anselm's thought sequence may be regarded as a formal system. This thought sequence as a computable function has denumerably many formulae. The limit of such a list is ω. In terms of Anselm's Definition the limit-concept ω denotes God.

Gödel’s result is saying that even if all provable truths are listed in a thought sequence there is at least one unproven truth outside the deductive system or sequence. This fits in with Anselm’s kind of thinking. Then, at least, one truth not provable in the system could be the truth about God.

What about the possibility of the non-proven truth that there is no God? Firstly, we recognize that it is God that we are talking about. The thought sequence is a sequence about God and is intended to approach God as closely as possible. The attributes of God are an interpretation or model of the thought sequence that makes the thought sequence come out true. (The model of a set of sentences is an interpretation under which they are all true (Blackburn (1996), p246.)

If it is claimed that, a particular truth does not denote God, one returns to the thought sequence until one does approach a limit point that meets the criteria to denote God, that is, that than which nothing greater can be thought. This is the power of Anselm’s Definition; that there is only one conceivable being that can meet the criteria. Given any lesser thought one always returns to the thought sequence to crank out more terms until a limit is obtained. The Limit is that than which nothing greater can be thought.

Gödel's Incompleteness Result does not require a limit, it simply requires a recursive set of formula complex enough to code metamathematical concepts such as proof and proof schema. Gödel's recursion technique allows him to build, step by step, symbol by symbol, formulae that can assert provability and non-provability. Anselm's Definition does not require a provability result but it can use a provability result. The provability result can be used to assert the existence of a true but unproven formula.

Gödel's technique of recursion can be used to generate the thought sequence. Gödel’s formula sequence or list need only be taken as far as the establishing of numeric coding of proofs and provable formulae statements. Anselm’s list is required as a limit, at least in the limit of the conceivable. But Gödel’s result is relevant because it is a limitative result on provability, placing a limit on the use of provability for a total set of formulae.

In conclusion, I would claim that Gödel’s Incompleteness Results are applicable to Anselm's Definition, particularly with respect to the relation between truth and provability. Any formal or deductive system that is consistent can be shown to have an undecidable truth. In the context of Anselm's Definition, I claim that the unproven truth can be seen to be the idea of God. This is conceptual evidence for the existence of God.