10 God and Diagonalisation

10.1 Anselm’s Definition and Cantor’s Diagonal Argument

Another important argument is Cantor’s Diagonal Argument. This argument recurs continually in mathematics and logic. I also intend to treat this as applicable to Anselm’s Definition. I will first of all give Cantor’s version of the argument, another version due to Finsler and then various illustrations of how the argument may be used. This argument will be used with direct reference to Anselm’s Definition.

Cantor proposed the Diagonal argument as a proof of the non-denumerability of the reals. That is, the real numbers (the decimals) cannot be put in a one-to-one correspondence with the positive integers. This can also be expressed by saying that the reals are dense, which is the property that between any two real numbers there is always another. This is not the case with the integers.

The argument is as follows.

Set up an array of real (decimals) numbers, say between 0 and 1. Here d_{ij} is a digit between 0 and 9 inclusive

$$
\begin{array}{lllll}
.\ d_{11} & d_{12} & d_{13} & d_{14} & \ldots \\
.\ d_{21} & d_{22} & d_{23} & d_{24} & \ldots \\
.\ d_{31} & d_{32} & d_{33} & d_{34} & \ldots \\
.\ d_{41} & d_{42} & d_{43} & d_{44} & \ldots \\
\quad . & & & &
\end{array}
$$

The claim for the reductio is made that all reals in the range are listed, a real number for each row of the matrix. Equivalently, one can say that all the reals between 0 and 1 can be enumerated or listed in a 1-1 correspondence with the natural numbers. Cantor argued that this cannot be the case because the diagonal number $.d_{11}d_{22}d_{33}d_{44} \ldots$ can be changed as follows:

$$\text{If } d_{ii} = 9 \text{ then } d_{ii} = 0$$
$$\neq 9 \text{ then } d_{ii} = d_{ii}+1.$$

This says if any digit of the diagonal number is 9 make it 0 and if it is not 9 add 1 to it.

This diagonal number cannot appear as a row of the array because it differs from the n^{th} number in the n^{th} digit. However, the diagonal number is a decimal between 0 and 1. The new diagonal number may be added to the list and the diagonalisation process repeated. But the original claim was that all reals were listed. Cantor has shown that they cannot be so listed.

10.2 Binary Form of the Diagonal Argument

Another form of the Diagonal argument is to use binary sequences. Can all binary sequences be listed (enumerated)?

M	1	2	3	4	.	.	.	N
	0	1	1	0	.	.	.	
	1	0	1	0				
Sequence	1	1	1	0				
	0	0	1	1				
.								
.								

Let b_{ij} be a binary digit (0 or 1) in the i^{th} row and the j^{th} column. A sequence will be a string of binary digits which indicates the presence or absence of an integer in the sequence. 0 means $n \in N$ does not occur in the sequence or row. 1 means $n \in N$ does occur in the sequence or row.

The claim is made that <u>all</u> binary sequences are listed or listable.

The diagonal b_{ii} can be changed as follows

$$b_{ii}' = 1 - b_{ii}$$

which is 1 if b_{ii} is 0 and 0 if b_{ii} is 1.

The new diagonal binary sequence, or anti-diagonal, is not on the list, that is, is not already a row, so the list is not complete. In fact, the list is never complete. Even if the new sequence is added to the list, the procedure can be repeated.

I will continue to use this binary form of the Diagonalisation Argument.

Generally, the diagonalisation argument can be expressed as a (square) matrix (finite or infinite) with rows and columns and a diagonal. The rows consist of objects under discussion and the columns consist of properties or values relevant to the objects. The diagonal is treated in an arbitrary way to obtain a constructible object, which cannot be on the object list because it differs from the n^{th} object in the n^{th} property or value.

The argument can then be taken several ways. Because we have constructed a new object (the new diagonal, or anti-diagonal) this object can go on the list. However, if the list was claimed to be already complete, (all values of a total function listed), then the claim that all values can be listed (computed) is false. So-called completed lists of total functions are not complete.

This kind of argument can be used to show that

1. There are more sets of integers than integers
2. There are more subsets in a set than elements of a set (Power set)
3. The number of predicates (where each line of the matrix represents a property that an integer does or does not have) on the integers is non-denumerable.
4. The number of functions on sets of integers is non-denumerable.
5. The set of algebraic numbers is denumerable and the set of transcendental numbers is non-denumerable. (Kleene (1952), p7).

The diagonal argument can be interpreted two ways. The first way is to say that the diagonal formed is just another item for the list of items being diagonalized. This means that if the claim is made that the list is complete what then is the status of the diagonal item? It is either another item for the list, hence the list is not complete, or an item of a different kind has been constructed.

The second way to use the diagonal argument is to claim that the list is complete and a new item has been constructed. This argument is used to demonstrate the existence of non-computable functions as follows. A computable function is programmable and is therefore a coding of a finite sequence of finite strings of a finite alphabet. Hence the computable functions are at most countably infinite (in a 1-1 correspondence with the positive integers). Diagonalizing over the countably infinite square matrix of computable functions we construct another function. Because the set of computable functions is complete (up to countably infinite), we have constructed something else, namely a non-computable function.

Presumably any argument that can be structured as a decidable, binary, square matrix is subject to diagonalisation.

Consider the following argument.

Let the matrix consist of conceivable objects where each row is a conceivable object and each column is the presence or absence of a conceivable property for that object. Diagonalizing over the matrix constructs an object not on the list. One can argue that the list of conceivable objects is uncountably infinite (dense: between any object one can always place another) so we have simply created another object for the list. However, if one argues that conceivable objects are not dense only countably infinite then one has diagonalized out an object not on the list of conceivable objects: an inconceivable object. But how can one have a constructible inconceivable object? I claim that this admirably fits Anselmic thinking and models religious language. This kind of argument is also consistent with the idea of a finite, or at most countably infinite, Creation.

Here I draw a distinction between language (what is conceivable) and the creation, which is the result of creative acts. Conceivable objects of language may be dense or non-denumerable. Created objects of the creation are finitely countable and not arbitrarily dense. A non-denumerable creation would be a continuous, or eternal creation and not a completed act. The creation consists of finitely many creative acts, completed in the creation by God.

This may be related to Anselm's argument as follows:

One can show that not all attributes (decidable predicates) can be listed (3 above). Similarly, if a proof is regarded as a decidable predicate (an object is proven or not proven) then not all proofs are listed. That is, there is a proof not on the list. This is consistent with the statement that the greater than is not an element of the sequence. It does not prove that the missing proof is the proof of the greater than. So generally with Anselm's argument: we are demonstrating the rationality and plausibility of the greater than without final conclusive proof that what is there is the greater than.

10.3 Summary of Arguments using Diagonalisation

The basic importance of the Diagonal Argument can be shown by the way in which many of the arguments we have been looking at can be expressed in terms of it. The diagonal argument appears in many forms.

Following is a summary of the main arguments and their Diagonal character.

The basic Diagonal argument pattern is as follows

1. LIST: Determine a list. This requires making a list of objects which forms a matrix with rows and columns. The major use in this thesis is of binary values (1 and 0), which describe how a decidable predicate does or does not describe an object in a specified cell position.

2. DIAGONAL: As the matrix is square there is a diagonal b_{ii}. This diagonal is changed by adding 1 to each b_{ii} so that a 1 becomes a 0 and a 0 becomes a 1, that is (b_{ii}'= b_{ii}–1). This new diagonal is often called an anti-diagonal. It is a constructed object not on the list for it differs from the n^{th} object (row) in the n^{th} position (column).

3. CONTRADICTION: A contradiction is obtained because the anti-diagonal is a constructed object not on the list and all objects are supposed to be listed. The notation for this is $b_{ii}' = b_{ki}$ because the constructed item should be on the list at, say, the kth row and $b_{ii}' \neq b_{ki}$ because by construction the anti-diagonal is not equal to any row.

4. RESOLUTION: A decision is made about whether the list is complete or not complete. If it is decided that the list is incomplete then the anti-diagonal is added to the list as a new row and the construction of a new diagonal is repeated. Alternatively, the list is regarded as complete and a new object, not belonging to the list, has been constructed.

Using this structure I will briefly outline some uses of Diagonalisation.

1. Cantor's original Diagonal Argument

1. LIST: Cantor creates a list of real numbers, $0<r_n<1$ where d_{ij} is the j^{th} digit in the i^{th} row.
2. DIAGONAL: He creates a diagonal real number d_{ii}, not on the list, where $d_{ii}=d_{ii}+1$ (mod 10).
3. CONTRADICTION: The list of reals is not complete when it is claimed to be complete.
4. RESOLUTION: The list is not complete therefore the reals are non-denumerable ($d_{ii} \neq d_{ki}$) for all k. This means the diagonal is never a member of the list so there are more reals than positive integers.

2. Binary version of the Diagonal Argument

1. LIST: Decidable predicates b_{ij} taking binary values (True = 1 False = 0).
2. DIAGONAL: Create anti-diagonal $b_{ii}' = b_{ii} - 1$.
3. CONTRADICTION: The anti-diagonal equals some row of the matrix, $b_{ii}' = b_{ki}$ but $b_{ii}' \neq b_{ki}$ by construction.
4. RESOLUTION: Using $b_{ii}' \neq b_{ki}$ for all k, we have a newly constructed object.

3. Diagonalisation by Boolos (Boolos & Jeffrey, p11)

1. LIST: Consider list L with entries L_n.

2. DIAGONAL: Construct a set D (Diagonal) such that $n \in D$ iff $n \notin L_n$.

3. CONTRADICTION: Claim $D = L_m$ for some m on the list. Then (from 2) $n \in D$ iff $n \notin D$.

4. RESOLUTION: Either L is not complete or D does not belong to L.

4. Diagonalisation and the non-computable functions

1. LIST: Enumeration L of functions $f_n(x)$ that are decidable or effectively computable.

2. DIAGONAL: Create the diagonal u(n)

$$u(n) = f_n(n)-1 \quad \text{if } f_n(n) \text{ defined}$$
$$= 0 \quad \text{if } f_n(n) \text{ not defined.}$$

3. CONTRADICTION: Function u(n) cannot be in L. If it is $u = f_m$ for some m giving $f_m(m) = u(m) = f_m(m)-1$ which is a contradiction.

4. RESOLUTION: We have constructed a non-computable function u. That is, a function not on the list of computable functions.

5. Diagonalisaton and the Halting Problem

1. LIST: List the computable functions f_x. Computable functions produce an output, halt and are defined. Non-computable functions do not halt and are not defined.

2. DIAGONAL: Construct the characteristic function g for f_x. This is the equivalent of the binary matrix.

$g(x,y) = 1$ if $f_x(y)$ is defined/halts

$= 0$ if $f_x(y)$ is not defined/does not halt

Hence construct diagonal d

$$d(x) = g(x,x)-1 \quad \text{if } g(x,x) \text{ is defined}$$
$$= \text{undefined} \quad \text{if } g(x,x) \text{ is not defined}$$

If g is constructible so is d.

3. CONTRADICTION: If d is computable then it is some g. Hence $g(x,x) = d(x) = g(x,x)-1$ which is a contradiction. So d is not computable.

4. RESOLUTION: If d is not computable then g is not computable. Therefore not all functions are computable (when defining computability by halting).

6. Gödel's Initial Version of Gödel Incompleteness Theorem (Gödel (1992), p39-40)

1. LIST: List of formulae $R_n(x)$ with one free variable x.

2. DIAGONAL: $n \in K \equiv \neg Bew\ (R_n(n))$ where K is the diagonal and ¬Bew means not provable.

3. CONTRADICTION: But k must be some formula on the list R_n. So $m \in K = R_m(m)$ for some m, giving
   $R_m(m) = m \in K \equiv \neg Bew\ (R_m(m))$.
   This is resolved as follows.

4. RESOLUTION: Either, if $R_m(m)$ is true it is not provable. Or, if $R_m(m)$ is provable it is true (after negating both sides).

7. Gödel Incompleteness Theorem by Finsler using Diagonalisation

1. LIST: Form two matrices for formulae F_i , one for truth (a_{ij}) and one for provability (b_{ij}).

   a_{ij} = 1 if $F_i(n)$ true
   = 0 if $F_i(n)$ false

   b_{ij} = 1 if $F_i(n)$ is proven
   = 0 if $F_i(n)$ is not proven.

2. DIAGONAL: Construct anti-diagonals $a_{ii}' = 1-a_{ii}$ and $b_{ii}' = 1-b_{ii}$.

3. CONTRADICTION:
   Claim there is a k such that $1-a_{ii} = a_{ki} = a_{ii}'$ and $1-b_{ii} = b_{ki} = b_{ii}'$

   Putting i = k gives $1-a_{ii} = a_{ii}$ and $1-b_{ii} = b_{ii}$ which are contradictory.

4. RESOLUTION: However, $1-a_{ii} = b_{ii}$ is not contradictory. That is, the anti-diagonal of a_{ij} may equal a row of b_{ij}. Possible binary values and interpretations of this are

   i) With $a_{ii} = 1$ gives $b_{ii} = 1-1 = 0$. That is, a_{ii} is true, and b_{ii} is not proven. However, with $a_{ii} = 0$ gives
   ii) $b_{ii} = 1-0 = 1$, which says a_{ii} is false and b_{ii} is true. This is ruled out because the set of formulae a_{ii} are true (consistent), so only i) applies. That is, the anti-diagonal (a formula) is true but not

proven.

8. Richard's Paradox as a Diagonalisation (Van Heijenoort, p142).

1. LIST All permutations of 26 letters of alphabet are lexicographically ordered. Cross out all permutations that are not definitions of numbers. List consists of all numbers that are defined by finitely many words. This forms a denumerably infinite set E.

2. DIAGONAL Create the diagonal D where the n^{th} number is changed in the n^{th} place in the number.

3. CONTRADICTION The diagonal D is a finitely constructible number not on the list that is supposed to contain all finitely described numbers.

4. RESOLUTION Richards claims that the set E is never complete therefore the diagonal is never fully defined.

9. The Liar and Diagonalisation

The Liar is not a diagonalisation but the diagonalisation process can show how the Liar can arise.

1. LIST: Create a list of decidable predicates $P_n(x)$.

2. DIAGONAL: A characteristic equation for P is as follows:

$$C_P(x) = 1 \text{ if } P(x) \text{ is true}$$
$$= 0 \text{ if } P(x) \text{ is false.}$$

If the decidable predicate P(x) is the truth predicate, T(x), we have the following characteristic equation

$$C_T(x) = 1 \text{ if } T(x) \text{ is true } \equiv \text{ if } (x \text{ is true}) \text{ is true}$$
$$= 0 \text{ if } T(x) \text{ is false } \equiv \text{ if } (x \text{ is true}) \text{ is false.}$$

This is the Liar because a true condition is called false. Therefore, this kind of characteristic equation for the truth predicate cannot be used and therefore diagonalisation cannot apply. However, if we use the partially defined characteristic equation

$$C_T(x) = 1 \quad \text{if } T(x) \text{ is true}$$
$$= \text{undefined otherwise.}$$

This does not always give a contradiction and such an equation is also not decidable (recursive).

10.4 Anselm's Definition reworked as a Diagonalisation

Can Anselm's Definition be reworked as a diagonalisation? In the following chapter I will explore the relevance of Incompleteness to Anselm's Definition. It seems to me that the thought sequence proposed by Anselm's Definition always has something missing, which is the idea that adequately describes God. With Anselm's Definition we are always approaching the most adequate idea of God but we never attain it. Is the ultimate idea of God attainable or are we always left with the fact of conceptual incompleteness of the unable-to-be-completed thought sequence? How are we to express this incompleteness? Diagonalisation offers a way to approach this situation, although it proves to have problems.

Consider a matrix that consists of all the created objects. I claim that this is a finite list, not an infinite list. The class (not set) of created objects is completed. An infinite class of created objects would involve a never ending creative sequence that not even God could complete, because from within the sequence there would always be another object to create. As far as the class of created objects (the creation) is concerned, we are (constructively) within it. Each row of the matrix represents a created object. This is a finite and completed list. The columns of the list of created objects consist of the properties of the objects. Each cell of the matrix is a 1 or a 0; a 1 if the object has the property and a 0 if the object does not have the property. So we have a very large, but finite, completed list of created objects with properties. Diagonalizing over this matrix produces a diagonal, which is a conceivable object because it has specified properties. This object may not exist. Because the list of created objects is complete, we cannot add it to the list of created objects, so we must have an uncreated object. No further diagonalizing can be done because there are no further created objects for the list. The question now becomes, what kind of object do we have specified by the diagonal? I cannot claim that it is God because the properties of the object are arbitrary. It may include properties such as hairy, round or red, none of which I would apply to God.

Can I possibly help the situation by arguing that I will only select objects that have properties that God has but not in a divine dimension? This would mean that we have a matrix that consists of created objects that have at least one property applicable to God. Each cell would contain a 0 if the property of the object was not divine and a 1 if the property was applicable to the divine. However, this arrangement would only supply certain attributes (properties) and not others. It would appear that the use of diagonalization can be is to give us an uncreated object, but we cannot guarantee the object obtained to be God. However, this does demonstrate the construction and possibility of a conceivable uncreated object.

10.5 Diagonalization and God

We have just shown the possibility of an uncreated object. Is this nonsense or can something be done with it? Religious imagery has the problem of describing the indescribable and conceiving the inconceivable. I want to explore the idea that diagonalisation allows us to constructively link the created and the uncreated.

The advantage of the diagonalisation on the created objects is that I have a means of constructively distinguishing the created from the uncreated, although the description I obtain does not help.

In his book, Beyond the Limits of Thought (Priest), Graham Priest offers some interesting reflections on contradictions that arise at the interface between the conceivable and inconceivable, the finite and the infinite. However, when he comes to the Anselm’s Ontological Argument, after a preliminary discussion, he seems to shrug his philosophical shoulders and wonder what it could mean (Priest, p63).

Nevertheless, I believe he has raised and provided in the book interesting ways to handle the idea of God. These ideas are based around Cantor's Diagonalisation Argument.

The diagonalisation argument can be summarised as follows, List, Diagonal, Contradiction and Resolution. These have the following meanings; a List consists of sets of items, a Diagonal item is created, a Contradiction ensues because the constructed diagonal is another item for the completed list, and a Resolution is made, where either the list is incomplete or a new object has been created.

Priest discusses diagonalisation. His notation is as follows:

x	list or set of items,
$\delta(x)$	diagonal obtained from the list,
$\delta(x) \in x$	claim the diagonal is an item of the list,
$\delta(x) \notin x$	claim the diagonal is not an item of the list.

Hence contradiction can be created.

The process of diagonalisation is about creating a contradiction. The contradiction is used to claim that the original list cannot be complete.

Priest develops his <u>Inclosure Schema</u> as follows (Priest, p147).

$\Omega=\{y : \phi(y)\}$	Ω is the set of all y such that ϕ
$\psi(\Omega)$	all Ω has property ψ
$x \subseteq \Omega$	x is a subset of Ω
$\psi(x)$	x has property ψ

x is diagonalized giving $\delta(x)$. Then either

$\delta(x) \in \Omega$ (the diagonal is a member of the complete list),

$\delta(x) \notin x$ (the diagonal is not a member of the list).

Priest recasts Russell's formula for paradox (Priest, p142). This can be set out as follows.

$\Omega = \{x : \phi(y)\}$ the class of all terms having property ϕ.

Treating x as a subset of Ω gives a diagonalisation δ with two possible interpretations.

a) $\delta(x) \in \Omega$

or b) $\delta(x) \notin x$

As this stands this is not a contradiction. However, in his search for contradiction, Priest can get a contradiction and the Russell paradox by setting $\Omega = x$ giving

a) $\delta(\Omega) \in \Omega$ closure,

b) $\delta(\Omega) \notin \Omega$ Transcendence.

This certainly gives contradiction because both cannot be true at the same time. However, rather than force a contradiction why not see a) and b) as two interpretations of the diagonalisation?

It is interesting that Priest attacks Cantor's Absolute Infinity 0_n by claiming that 0_n+1 simply continues the ordinal sequence and that there is nothing absolute about it. But this is part of the problem of dealing with completed infinities which Priest himself insists on doing with his use of the Domain Principle. Priest cannot get the contradiction he seeks without completed infinities. It would seem that Cantor's use of Absolute Infinity is an attempt to try and stop what he, Cantor, had started. That is, to get some completed conceptual totality to the ordinal sequence.

Priest uses the Domain Principle (Priest, p137) to give him his completed list. The Domain Principle claims:

> To every potential infinity there is a corresponding actual infinity.

So armed with the Domain Principle on the one hand, which gives him any completed infinite list he wants, and on the other hand, the Inclosure Schema, which by diagonalisation gives him an item not on the list, Priest can always obtain contradictions at the limits of any completed infinite list he requires.

Priest pursues contradiction for other reasons not relevant to this discussion. I wish to point out that his cultivation of contradiction may have a use in the context of Anselm’s Definition.

Contradiction surfaces immediately in religious language. How does one talk conceivably about the inconceivable, finitely about the infinite, effably about the ineffable? By his Inclosure Schema Priest has shown how to both hold and release contradiction. This is Closure versus Transcendence. Rather than force the contradiction, why not use the idea of God to hold Closure and Transcendence in some creative tension?

Closure = we talk about God;

Transcendence = we cannot talk about God.

Both these ways of speaking relate to God and have viable religious significance.

Anselm’s Definition uses transcendence. Anselm directs our thought to something beyond our thought.

So Priest in using Diagonalisation in his Inclosure Schema shows us a recipe, for creating contradiction which we tolerate with the concept of God (not the faceless wall of contradiction). I use this application of diagonalisation to show how the concept of God can be used to tolerate a contradictory position in a workable way. God, as a concept, is the only way I know how to combine opposites, not necessarily as contradictories, but as ‘constructible’ elements.